PC0352: X–W and X–Z are incompatible

GPT-6.1 Sol and Codex subagents

November 1, 2026 — prepared for independent review

Theorem. No nonempty finite square-lattice region can be tiled entirely by X pentominoes and also entirely by W pentominoes. The same holds for X and Z. All rotations and reflections are permitted. Holes and disconnected regions are permitted.

This is a computer-assisted proof: a finite inductive invariant is supplied explicitly, its finite conditions have independently checked SAT certificates, and the reduction from arbitrary finite regions to those conditions is below. It is not an extrapolation from bounded construction searches.

Reproduce the finite checks

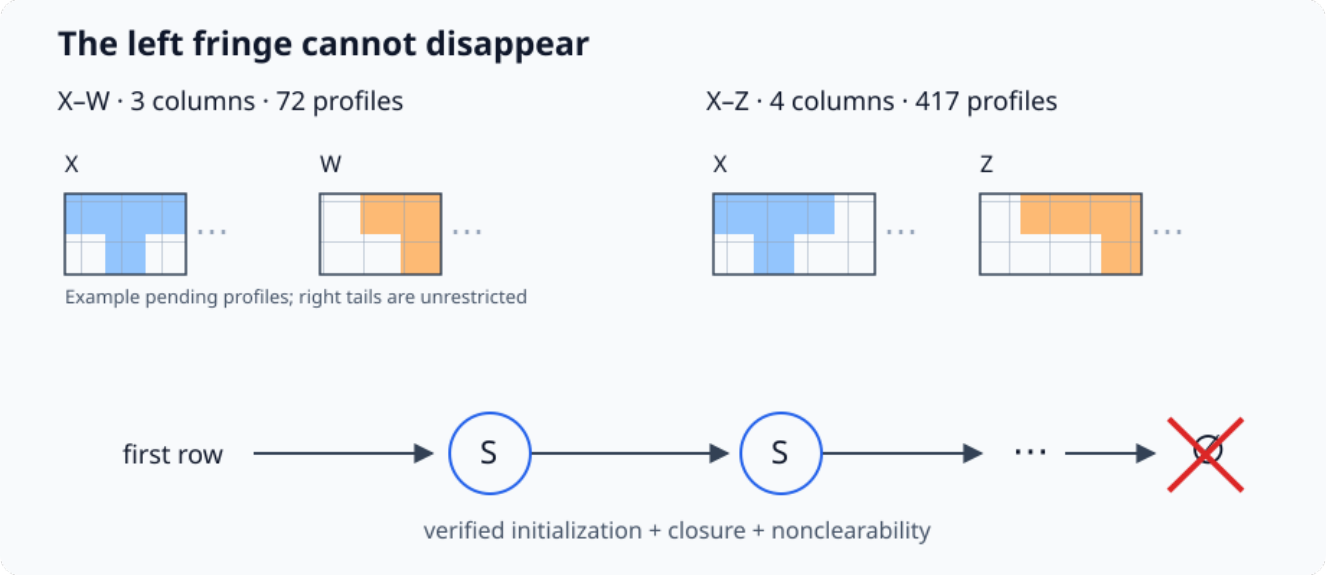
From this directory, using Python with PySAT, Kissat, and `drat-trim`:

```
python fringe_verify.py certificates/xw.json --output certificates/xw
python fringe_verify.py certificates/xz.json --output certificates/xz
python -m unittest -v test_search test_frontier test_fringe_verify
```

Each verifier invocation constructs all its CNF formulas anew, solves them, and checks each DRAT proof with the separate `drat-trim` executable. It fails unless every formula is UNSAT and every proof is verified. The stored CNFs, DRAT proofs, checker outputs, and `results.json` files are in the two output directories. Total: ten cases for X–W and twelve for X–Z.

`fringe_verify.py` does not import any placement generator, frontier transition, or invariant-discovery code. It defines the tiles independently in ASCII, generates all distinct D4 images, and uses pairwise nonoverlap clauses and Boolean cell-coverage variables. Its local windows differ from those used by the original CP-SAT discovery/checking implementation in `invariants.py`.

Positive-control tests deliberately remove an attainable initial profile from each certificate and obtain geometrically valid SAT counterexamples. The models therefore do not pass merely because their geometry is inconsistent.



1. Row frontiers

Increase row coordinates downward. At a cut just before row zero, call tiles whose topmost row is negative **old** and tiles starting in row zero **new**. All X, W, and Z orientations have height three and horizontal coordinate span two. Old tiles crossing the cut consequently start in rows -2 or -1 and have pending cells only in rows zero and one. Tiles ending above the cut are irrelevant to the next step.

For each of the two tilings, record its old pending occupancy in these two rows. A row transition chooses new tiles, without same-family overlaps, so that both tilings cover exactly the same cells in row zero, each zero or once. The next frontier is their combined old/new occupancy in rows one and two. Cells may be empty anywhere, so this permits arbitrary holes.

For a nonempty frontier let L be the leftmost column occupied by either family in either pending row. Retain only its first K columns. Its **profile** is

`(X row0, X row1, counterpart row0, counterpart row1),`

four K-bit integers, with the least significant bit representing column L. Everything to the right of this fringe is forgotten, not required to be empty.

The supplied finite sets are:

Counterpart	K	Allowed profiles S	Checked shifts
W	3	72, in `certificates/xw.json`	-4 through +4
Z	4	417, in `certificates/xz.json`	-5 through +5

How these sets were discovered is not a premise of the proof. Any literal sets satisfying the following conditions would suffice.

2. Three finite conditions

Initialization. Starting with no old tiles, every nonempty next frontier has its profile in S.

Nonclearability. For every `(a,b,c,d)` in S,

`b != 0 or d != 0 or a != c.`

This is checked directly on the literal tables. There is either an old second-row pending cell, or an old first-row mismatch between the families, within the fringe.

Bounded-shift closure. If an old frontier's profile belongs to S, a row transition whose new leftmost pending column T satisfies

`-K-1 <= T-L <= K+1`

has its new profile in S.

The SAT checks seek counterexamples to initialization and closure and prove them absent. They allow unmatched past coverage and arbitrary unexamined right context. Same-family nonoverlap is imposed on all selected tile cells, including past rows.

3. Why the finite windows cover arbitrary right context

Here are the exact windows used by the independent verifier. Normalize the old left edge to L=10 and use columns 0 through 27. For bounded shifts, T ranges from 6 to 14 for W, or 5 to 15 for Z.

Enumerate every old crossing tile starting in row -2 or -1 and every new tile starting in row zero that fits in this window. Match the two coverages in row zero only at columns 0 through 25. Constrain old pending cells before L to be empty, constrain the source fringe to S, and require the target fringe to lie outside S, with no target pending cells before T.

Any actual full row transition violating closure would give a satisfying assignment to this finite model. To see this, retain every actual old/new tile meeting a matched row-zero cell or a source/target fringe cell; discard the others.

- Every old retained tile has some pending cell at or right of L, so its leftmost cell is at least L-2=8.
- Every new tile has a future cell at or right of T, so its leftmost cell is at least T-2, which is at least three.
- A tile meeting a matched row-zero cell at column at most 25 extends no farther right than column 27.
- Fringe cells extend no farther right than column 18, so tiles retained only for those cells extend no farther right than column 20.

Thus every retained tile fits. Retention preserves coverage at every checked cell: all actual tiles meeting such a cell were retained. There is no recursive completion requirement at the right edge because equality is deliberately not checked in the last two columns. Removing other tiles cannot introduce overlaps. Past equality was not required, and omitting it only relaxes the model. Therefore UNSAT excludes a counterexample of any width.

For initialization, normalize the new pending left edge to 10 and use only new tiles. Match row zero at columns 0 through K+11, inclusive; leave the rest unconstrained. Retain the new tiles meeting those cells or the target fringe. Their leftmost coordinates are at least eight, and their rightmost coordinates are at most K+13, hence they also fit the window. The initial SAT model therefore covers all actual initial profiles, regardless of width.

4. Why all shifts, and empty targets, are covered

Let an old frontier have profile in S.

If its second pending row contains a fringe cell, that cell persists into the next frontier. Otherwise nonclearability gives a first-row mismatch at some column x <= L+K-1. To match row zero, the missing family must place a new tile covering x. Every such tile has a future cell and horizontal span at most two, so some future cell is at column at most x+2. Consequently the next frontier is nonempty and

`T <= L+K+1.`

If T >= L-K-1, the verified bounded-shift condition applies.

Otherwise T <= L-K-2. The new fringe occupies columns T through T+K-1. After translating its left edge to 10, the initialization test examines the original current-row interval [T-10,T+K+1], entirely to the left of L. No old pending cells occur in that range. Delete the old tiles and retain the new tiles meeting this current-row range or the new fringe. Those current-row cells still match, and all relevant future occupancy is unchanged. After translation this is an instance of the initialization model, so its profile also belongs to S. This covers arbitrarily distant new components starting to the left; no bound on the jump is assumed.

Thus every actual transition from a profile in S has a nonempty next frontier whose profile is again in S.

5. Contradiction with finiteness

Suppose a nonempty finite common region existed. At its first occupied row, all tiles are new. At least one X tile starts there and has future cells, so initialization puts its nonempty pending frontier in S. By induction, pending occupancy can never vanish. But after processing the last occupied row of a finite region, no pending cells remain. Contradiction.

The argument applies separately to the two certified sets, proving both incompatibilities. It never assumes connectivity, absence of holes, a bound on area or dimensions, or a restricted subset of rotations/reflections.