Sylver Coinage g=2 positions in 14

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You asked me to find the fourteenth man
for your expedition, and I chose Mr. Baggins.
– Tolkien, *The Hobbit*

1 Introduction

In my last article, I detailed solutions to all g=2 positions in 8 and 10. In this document, I continue the analysis, completely solving 14.

There are four winning replies to 14: 7, 8, 10, and 26. The highest move not eliminated by 8 is 34, therefore all positions without any other numbers 34 or below have 8 as a winning move. Despite the smaller bound on the second smallest number in a position, there are slightly more positions to solve in 14 than in 10. This is due to the greater branching potential for positions in 14.

A good reply to every position (or $\mathcal{P}$ if it is a $\mathcal{P}$-Position) will be listed in a similar way to the previous article.

2 {14,16}

{14,16} has 8 as a winning move, as do all derived positions not containing 18, 20, 26, or 34

2.1 {14,16,18}

{14,16,18} has 10 as a winning move, as do all derived positions not containing 22.

{14,16,18,22}	**20**	**{}**
24,26	31	9
24	$\mathcal{P}$	20
26	11	33
{}	24	45

2.2 {14,16,20}

{14,16,20}	**22**	**38**	**{}**
24,26	17	X	47
24	25	X	89
26	31	43	$\mathcal{P}$
{}	41	37	26

2.3 {14,16,26}

{14,16,26}	**24,34**	**24**	**34**	**{}**
22	9	9	51	85
36,38	X	X	9	9
36	13	33	137	41
38	X	X	9	9
{}	373	63	65	23

2.4 {14,16,34}

{14,16,34}	**36,40**	**36**	**40**	**{}**
22,24	X	X	X	35
22	X	X	19	265
24	X	13	X	39
38	25	19	89	79
52,54	X	X	X	67
52	X	X	9	133
54	X	19	X	25
{}	9	19	9	77

3 {14,18}

{14,18} has 5 as a winning move, and the only moves not eliminated by 5 are 22 and 26. However, {14,18,22} has 10 as a winning move, as do all derived positions not containing 26.

3.1 {14,18,26}

{14,18,26}	**22,24**	**22,38**	**22**	**24**	**38**	**{}**
20,30	15	X	57	11	X	51
20	15	X	25	19	X	77
30,34	81	17	$\mathcal{P}$	9	11	111
30	$\mathcal{P}$	24	24	55	43	9
34	31	97	30	7	87	9
48	X	X	X	X	13	23
{}	30	57	15	29	501	119

4 {14,20}

{14,20} has 10 as a winning move, as does any position not containing 22, 26, 32, 36, or 46

4.1 {14,20,22}

{14,20,22}	**24**	**38**	**52**	**{}**
26,30,32	39	7	X	$\mathcal{P}$
26,30	135	13	X	11
26,32	13	19	X	19
26	41	7	X	71
30,32	15	35	X	65
30,46	X	13	X	15
30	47	13	X	57
32	15	45	X	11
46	X	55	15	15
{}	53	213	25	47

4.2 {14,20,26}

{14,20,26} has 16 as a winning move. The only moves not eliminated by 16 are 24 and 38.

{14,20,24,26}	**30,32**	**30**	**32**	**{}**
36	29	7	23	13
{}	645	29	77	27

{14,20,26,38}	**30,32**	**30**	**32,44**	**32**	**44**	**{}**
36	19	21	11	11	47	97
50	X	X	11	11	51	29
{}	27	37	11	11	59	111

4.3 {14,20,32}

{14,20,32}	**36**	**50**	**{}**
24,30	19	X	127
24	7	25	25
30,38	7	X	43
30	89	X	25
38,44	23	17	105
38	27	7	75
44	11	11	11
58	11	11	11
{}	11	11	11

4.4 {14,20,36}

{14,20,36}	**30**	**44**	**58**	**{}**
24,46	29	X	X	13
24	$\mathcal{P}$	X	X	13
38,46	51	35	X	65
38	159	27	X	55
46,52	39	7	35	133
46	979	181	25	83
52	7	47	61	65
66	X	7	41	117
{}	53	$\mathcal{P}$	7	45

4.5 {14,20,46}

{14,20,46}	**30**	**44**	**58**	**72**	**{}**
24,50	37	X	X	X	17
24	36	X	X	X	167
38,50	X	7	X	X	39
38,64	X	25	X	X	87
38	35	25	X	X	359
50,52	X	33	25	X	53
50	X	113	29	7	29
52,64	X	X	25	X	13
52,78	X	X	X	X	51
52	29	41	25	X	$\mathcal{P}$
64	X	X	7	13	13
78	X	X	X	81	7
{}	57	29	75	125	49

5 {14,22}

{14,22} has 8 (and 9) as winning moves. The only moves not eliminated by 8 are 26 and 34.

5.1 {14,22,26}

{14,22,26}	**30,34**	**30**	**34**	**{}**
24,32	41	129	39	19
24	13	11	11	13
32,38	19	13	195	25
32	19	29	13	33
38,46	15	33	15	29
38	$\mathcal{P}$	19	41	11
46	35	77	33	11
60	X	X	X	29
{}	15	19	15	931

5.2 {14,22,34}

{14,22,30,34}	**32,40**	**32**	**40,46**	**40**	**46,54**	**46**	**54**	**{}**
24	11	191	X	13	X	X	7	11
38	57	59	49	155	73	$\mathcal{P}$	45	35
{}	7	41	17	11	25	55	17	111

{14,22,34}	**32**	**46**	**60**	**74**	**{}**
24,40	251	X	X	X	63
24,54	X	X	X	X	77
24	29	X	X	X	$\mathcal{P}$
38,40	35	101	X	X	57
38,54	X	35	X	X	47
38	73	155	X	X	24
40,52	13	29	61	X	1231
40	13	95	117	X	55
52,54	X	57	45	X	65
52	13	53	41	X	24
54	X	45	59	71	$\mathcal{P}$
{}	13	35	17	29	24

6 {14,24}

{14,24} has 8 and 10 as winning moves. The only moves not eliminated by 8 are 26 and 34.

6.1 {14,24,26}

{14,24,26}	**30**	**44**	**58**	**{}**
32,34,36	9	117	X	45
32,34	9	37	X	47
32,36	7	155	X	77
32	15	15	X	15
34,36,46	9	11	X	11
34,36	9	11	X	11
34,46	9	11	X	11
34	9	11	X	11
36,46	11	39	23	$\mathcal{P}$
36	11	47	23	46
46	11	25	31	36
60	X	23	41	181
{}	11	23	75	7

6.2 {14,24,34}

{14,24,34} has 10 as a winning move. The only moves not eliminated by 10 are 32, 36 and 46.

{14,24,32,34}	**30,40**	**30**	**40,44**	**40**	**44,54**	**44**	**54**	**{}**
36	11	97	53	43	83	$\mathcal{P}$	75	99
50	11	59	21	127	69	31	31	175
{}	11	21	61	35	211	67	23	59

{14,24,34,36}	**30,40**	**30**	**40,44**	**40**	**44,54**	**44**	**54**	**{}**
46	59	11	47	49	15	15	15	15
{}	83	29	59	19	35	37	37	22

{14,24,34,46}	**30,50**	**30**	**44,50**	**44,64**	**44**	**50**	**64**	**{}**
40	83	35	23	X	83	37	X	45
54	X	X	63	19	2875	73	79	87
{}	11	11	25	22	22	43	22	22

The only remaining position is {14,24,34,46,78}: 17

7 {14,26}

{14,26} is a $\mathcal{P}$-Position. It is therefore the longest case in this analysis, as there could not be any elimination like the previous cases.

7.1 {14,26,30}

{14,26,30} has winning move 9, as do all derived positions not containing 34 or 38.

{14,26,30,34}	**36,38**	**36**	**38,50**	**38**	**50**	**{}**
32	23	15	7	25	37	31
46	31	19	39	131	13	85
{}	221	101	29	39	55	7

{14,26,30,38}	**36,48**	**36**	**48,50**	**48**	**50,62**	**50**	**62**	**{}**
32	15	999	31	59	X	57	X	7
46	93	27	13	69	111	$\mathcal{P}$	31	31
{}	45	7	57	29	47	71	45	41

7.2 {14,26,32}

{14,26,32,76} has winning move 39, all other positions are in the following table:

{14,26,32}	**34,44**	**34**	**44,48**	**44,62**	**44**	**48**	**62**	**{}**
36,38	31	47	15	X	133	35	X	7
36	9	15	55	X	29	53	X	93
38,50	53	103	27	115	$\mathcal{P}$	19	27	41
38	117	33	169	19	103	121	41	27
50	9	9	97	45	155	19	23	51
{}	9	9	39	31	19	55	97	1513

7.3 {14,26,34}

{14,26,34} has winning move 9, as do all derived positions not containing 38.

{14,26,34,38}	**44,46**	**44**	**46,58**	**46**	**58**	**{}**
36	77	29	29	29	53	139
50	35	59	13	119	37	53
64	47	167	19	31	13	133

7.4 {14,26,36}

If the position contains 38 or 46, but not both, then 11 is a good move. The table for both 38 and 46 is:

{14,26,36,38,46}	**48**	**{}**
44	19	67
58	7	13
{}	25	23

The table for if neither is in the position is:

{14,26,36}	**48,60**	**48**	**60**	**74**	**{}**
44	61	53	53	161	$\mathcal{P}$
58	79	103	31	35	44
{}	27	93	13	37	37

7.5 {14,26,38}

{14,26,38} has winning move 11, as do all derived positions not containing 46.

{14,26,38,46}	**44,50**	**44**	**50,58**	**50**	**58**	**{}**
48	41	121	25	43	25	19
62	33	45	23	25	27	25
{}	131	15	85	7	15	15

7.6 {14,26,44}

Any position containing 46 has winning move 11, the other positions are in the following table:

{14,26,44}	**50,60**	**50,74**	**50**	**60,64**	**60**	**64,74**	**64**	**74**	**{}**
48	35	X	55	31	67	X	37	X	71
62	13	25	253	67	59	25	27	25	7
76	X	X	X	65	15	7	41	15	15
90	X	X	X	X	51	X	X	67	127
{}	45	13	39	291	$\mathcal{P}$	19	63	37	37

7.7 {14,26,46}

All positions in {14,26,46} move to 11.

7.8 {14,26,48}

{14,26,48}	50,60	50	60,64	60	64	{}
58	49	13	75	33	127	35
72	51	111	7	39	61	33
86	X	69	X	X	101	41
{}	55	$\mathcal{P}$	79	89	50	50

7.9 {14,26,50}

{14,26,50}	60,62	60	62,74	62	74	88	{}
58	67	201	31	39	31	25	2655
72	19	81	81	31	25	31	31
86	X	X	73	65	39	75	53
{}	509	27	48	48	45	41	48

7.10 {14,26,58}

{14,26,58}	62,64	62	64,76	64	76	90	{}
60	7	201	35	55	15	13	297
74	13	61	185	37	7	99	357
88	X	X	25	37	25	25	25
102	X	X	X	37	X	7	39
{}	19	19	785	37	15	105	7

7.11 {14,26,60}

{14,26,60}	62,64	62	64,76	64	76	90	{}
72	27	243	75	165	15	103	41
{}	135	57	129	39	15	49	51

7.12 {14,26,62}

{14,26,62}	72,74	72	74,86	74	86	100	{}
64	35	59	87	69	51	97	2499
{}	187	25	75	81	103	39	279

7.13 {14,26,64}

{14,26,64}	74,76	74	76,88	76	88	102	{}
72	25	25	55	109	87	39	301
86	85	29	59	19	115	19	19
100	X	X	73	27	185	61	95
114	X	X	X	115	X	71	27
{}	71	29	703,749	$\mathcal{P}$	57	76	121

7.14 {14,26,72}

{14,26,72}	74	88	102	116	{}
76	25	13	X	X	15
90	25	39	13	X	249
{}	25	27	111	59	641

7.15 {14,26,74}

{14,26,74}	76	90	{}
86	13	149	67
{}	15	75	123

7.16 {14,26,76}

All positions in {14,26,76} have 15 as a winning move.

7.17 {14,26}

{14,26}	86	100	114	128	{}
88,90	13	89	X	X	35
88	175	35	X	X	79
90,102	19	111	41	X	883
90	19	83	123	295	643
102	19	277	35	X	41
116	19	321	41	35	621
{}	19	413	69	39	$\mathcal{P}$

And {14,26,142} [27

8 {14,30}

{14,30} has winning move 8 and 10. The only move not eliminated by 8 is 34, so this case reduces to {14,30,34}. {14,30,34} has winning move 10, as do all other derived positions not containing 32, 36, or 46.

{14,30,32,34}	**38,40**	**38,54**	**38**	**40,52**	**40**	**52,54**	**52**	**54**	**{}**
36	15	9	9	19	19	81	125	73	25
50	31	9	9	43	91	17	51	29	17
{}	235	9	9	67	21	91	15	49	579

{14,30,34,36}	**40,46**	**40**	**46,54**	**46**	**54**	**{}**
38	53	21	9	9	9	9
52	41	11	47	83	25	17
{}	139	11	53	23	89	23

{14,30,34,46}	**40,50**	**40**	**50,54**	**50**	**54**	**{}**
38	59	15	9	9	9	9
52	13	25	13	13	39	19
66	13	59	13	13	45	19
{}	13	109	13	13	21	19

9 {14,32}

{14,32} has 8 as a winning move, as does every derived position not containing 34.

{14,32,34,36}	**38,40**	**38,54**	**38**	**40,52**	**40**	**52,54**	**52**	**54**	**{}**
44	15	7	63	43	29	23	45	165	139
58	15	121	7	25	71	93	63	83	49
{}	15	49	131	147	55	31	89	25	43

{14,32,34,50,86}: 35, {14,32,34,86}: 23

{14,32,34,50}	**38,40**	**38,54**	**38**	**40,52**	**40**	**52,54**	**52**	**54**	**{}**
44	9	15	89	17	69	35	59	107	67
58	99	15	17	21	57	25	87	35	69
72	X	X	X	X	X	49	61	67	45
{}	379	15	143	57	133	875	205	183	$\mathcal{P}$

{14,32,34,50}	**38,40**	**38,54**	**38**	**40,52**	**40**	**52,54**	**52**	**54**	**{}**
44	9	57	7	123	65	45	15	35	43
58	53	7	203	61	29	25	15	17	37
72	X	X	X	X	X	79	15	45	17
{}	61	85	31	23	29	93	15	73	50

10 {14,34}

{14,34} has 10 as a winning move, as does every derived position not containing 36 or 46.

10.1 {14,34,36}

{14,34,36,38}	**44,46**	**44,60**	**44**	**46,58**	**46**	**58,60**	**58**	**60**	**{}**
40	9	9	9	7	475	23	23	7	43
54	11	25	101	11	11	101	25	17	21
{}	7	11	11	47	39	11	11	11	7

{14,34,36,44} has 13 as a winning move, as does every derived position not containing 46.

{14,34,36,44,46}	**40**	**54**	**{}**
52	37	35	47
66	17	23	25
{}	$\mathcal{P}$	103	65

{14,34,36,40}	**46,52**	**46,66**	**46**	**52,60**	**52**	**60,66**	**60**	**66**	**{}**
58	19	19	19	15	15	39	31	17	103
{}	51	97	29	15	15	25	91	31	105

{14,34,36}	**46,54**	**46**	**54,60**	**54,74**	**54**	**60**	**74**	**88**	**{}**
52,58	55	45	39	87	??	21	31	X	247
52	103	17	191	53	17	43	85	X	31
58,66	29	51	17	201	67	125	55	21	3245
58,80	X	X	45	189	21	67	47	65	121
58	43	779	805,827	39	125	217	99	25	21
66	29	49	103	17	199	65	93	141	487
80	X	X	55	43	31	111	21	121	75
94	X	X	X	67	59	X	47	49	17
{}	41	109	257	$\mathcal{P}$	74	59	47	61	41

10.2 {14,34,46}

In {14,34,38,46}, if 54 is in the position, it will have winning move 11.

{14,34,38,46}	**40,50**	**40,64**	**40**	**50**	**64**	**78**	**{}**
44	29	15	15	21	55	X	201
58	13	15	15	13	49	85	331
{}	45	15	15	7	113	49	21

{14,34,40,46}	**50,52**	**50,66**	**50**	**52,64**	**52,78**	**52**	**64,66**	**64**	**66,78**	**66**	**78**	**{}**
44	19	19	19	17	X	23	63	79	X	127	X	57
58	41	17	115	111	59	$\mathcal{P}$	77	57	65	117	297	53
72	61	31	21	55	95	105	85	69	75	79	153	83
{}	1157	53	65	59	93	31	31	63	51	65	97	159

{14,34,44,46}	**52,54**	**52**	**54,66**	**54**	**66**	**{}**
50	31	23	55	97	57	61
64	23	37	57	249	71	69
{}	17	71	51	121	179	79

{14,34,46,50}	**52,54**	**52**	**54,66**	**54**	**66**	**{}**
58	17	65	31	23	37	21
72	79	41	145	61	109	147
86	X	X	89	103	239	39
{}	43	49	$\mathcal{P}$	66	32	32

{14,34,46}	**52,64**	**52,78**	**52**	**64,66**	**64**	**66,78**	**66**	**78**	**{}**
54,58	105	45	40	71	39	183	321	37	29
54,72	43	75	191	21	83	85	57	17	29
54,86	X	X	X	43	67	55	233	53	29
54	65	201	53	17	109	50	50	95	29
58	171	103	40	39	247	29	31	49	111
72	141	97	49	57	31	29	59	63	57
86	X	X	X	63	139	29	113	195	103
100	X	X	X	X	79	X	X	73	45
{}	21	45	109	167	4159	29	71	447	317

All higher positions in {14,34} have 10 as a winning move, and all other positions in 14 have 8 as a winning move.

11 Conclusion and further questions

11.1 Analysis of higher values

One may ask about higher valued positions. 12 has been analysed up to and including {12,26}, but due to the lack of a g=2 winning reply to {12} there are a potentially infinite number of positions to analyse. 16 has similar issues, having no known winning move at all. The only other initial move that (given known replies) be fully analysed is 26. However, there is so much branching that one could only hope to approach small subcases of it.

I will continue to research the positions in g=2, with the goal to catalogue the winning moves for a range of positions with g=2, starting with those in {16,24}.

There has been progress on the case of g=3, with new $\mathcal{P}$-Positions having been found. There is a possibility for further analysis there. However, many long positions have no known winning move. Unfortunately, no analogue for the Pattern Theorem exists for g=3, leaving many ambiguous positions, the simplest of which being {12,15,21}.

The positions with g=4 run into similar trouble. The only known $\mathcal{P}$-Position with g=4 is {8,12}, which by a pairing argument can be shown to have no good reply. Various long positions show some potential to be $\mathcal{P}$, such as {16,20,28}. However, no other short positions have been found to be $\mathcal{P}$. Further analysis could be done into g=4 short positions to find one with similar pairing behaviour as {8,12}.